On the ideal case of a conjecture of Huneke and Wiegand

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§1 Introduction

For a moment

- R an integral domain
- M, N finitely generated torsion-free R-modules

Recall that M is called torsion-free, if the natural map

$$0 \to M \to M \otimes_R \mathrm{Q}(R)$$

is injective.

Question

When is the tensor product $M \otimes_R N$ torsion-free?

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is injective.

Question

When is the tensor product $M \otimes_R N$ torsion-free?

Let $(-)^* = \mathrm{Hom}_R(-, R)$ be the algebraic dual.

Conjecture 1.1 (Huneke-Wiegand, 1994)

Let $(R, \mathfrak{m})$ be a Gorenstein local domain with $\dim R = 1$, M a finitely generated torsion-free R-module. If $M \otimes_R M^*$ is torsion-free, then M is free.

Theorem 1.2 (Auslander, 1961)

Let R be a Noetherian normal domain, M a finitely generated R-module. Then M is projective if and only if $M \otimes_R M^*$ is reflexive.

Theorem 1.3 (Huneke-Wiegand, 1994)

Let R be a hypersurface domain, M, N finitely generated R-modules. If $M \otimes_R N$ is torsion-free, then either M or N is free.

Theorem 1.4 (Celikbas, 2011)

Let R be a complete intersection domain, M a finitely generated torsion-free R-module with bounded Betti numbers. If $M \otimes_R M^*$ is torsion-free, then M is free.

For a commutative Noetherian local ring R, we define

(HWC) For every finitely generated torsion-free R-module M, if $M \otimes_R M^*$ is reflexive, then M is free.

(ARC) For every finitely generated R-module M, if $\mathrm{Ext}_R^{>0}(M, M \oplus R) = (0)$, then M is free.

Theorem 1.5 (Celikbas-Dao, Cekilbas-Takahashi, Huneke-Wiegand)

Consider the following conditions.

(1) (HWC) *holds for all Gorenstein local domains.*

(2) (HWC) *holds for all one-dimensional Gorenstein local domains.*

(3) (ARC) *holds for all Gorenstein local domains.*

Then the implications (1) $\Longleftrightarrow$ (2) $\Longrightarrow$ (3) *hold.*

Conjecture 1.6

Let $(R, \mathfrak{m})$ be a Gorenstein local domain with $\dim R = 1$, I an ideal of R. If $I \otimes_R I^*$ is torsion-free, then I is principal.

Fact 1.7

Conjecture 1.6 holds for the following cases.

(1) $\mathrm{e}(R) \leq 6$ (Goto-Takahashi-T-Truong, 2015)

(2) $R = k[[H]]$, I *monomial,* $\mathrm{e}(R) \leq 7$ (Goto-Takahashi-T-Truong, 2015)

(3) $\mathrm{e}(R) \leq 8$ (Huneke-Iyengar-Wiegand, 2018)

(4) $I \cong$ *trace ideal* (Lindo, 2017)

The main result of this talk is stated as follows.

Theorem 1.8 (Celikbas-Goto-Takahashi-T, 2018)

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with $\dim R = 1$, I an $\mathfrak{m}$-primary ideal of R. Suppose that I is weakly $\mathfrak{m}$-full, that is, $\mathfrak{m}I : \mathfrak{m} = I$. If $I \otimes_R I^*$ is torsion-free, then I is principal, and hence R is a DVR.

Contents

§2 Weakly $\mathfrak{m}$-full ideals

Throughout, let

- $(R, \mathfrak{m})$ be a Noetherian local ring
- I an ideal of R

Definition 2.1

(1) I is called *$\mathfrak{m}$-full*, if $\mathfrak{m}I : x = I$ for $\exists\, x \in \mathfrak{m}$ (Rees)

(2) I is called *weakly $\mathfrak{m}$-full*, if $\mathfrak{m}I : \mathfrak{m} = I$ (Celikbas-Iima-Sadeghi-Takahashi)

Fact 2.2 (Goto, Rees)

Suppose $|R/\mathfrak{m}| = \infty$. If $I = \overline{I}$, then $I = \sqrt{(0)}$ or I is $\mathfrak{m}$-full.

Remark 2.3

The implication

$$I \text{ is } \mathfrak{m}\text{-full} \implies I = \overline{I}$$

does not hold.

Example 2.4

Let $R = k[[X, Y]]$ be the formal power series ring over a field k. Then

$$I = (X^3, X^2Y^3, XY^4, Y^5)$$

is $\mathfrak{m}$-full, but $I \neq \overline{I}$.

Example 2.5

Let J be an ideal of R and set $I = J : \mathfrak{m}$. Then I is a weakly $\mathfrak{m}$-full ideal.

Remark 2.6

The implication

$$I \text{ is weakly } \mathfrak{m}\text{-full} \implies I \text{ is } \mathfrak{m}\text{-full}$$

does not hold.

Proposition 2.7

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring, Q a parameter ideal of R. We set $I = Q : \mathfrak{m}$. If

$$\mu_R(\mathfrak{m}) > \dim R + \mathrm{r}(R),$$

then I is not $\mathfrak{m}$-full.

Proof.

Note that R is not regular. Then $\mathfrak{m}I = \mathfrak{m}Q$, so that

$$\mu_R(I) = \dim R + \mathrm{r}(R)$$

because $\ell_R(I/Q) = \mathrm{r}(R)$. If I is $\mathfrak{m}$-full, then

$$\mu_R(I) \geq \mu_R(\mathfrak{m})$$

which makes a contradiction. □

Example 2.8

Let k be a field and set $R = k[[t^5, t^6, t^7, t^9]]$. Then

$$I = (t^5, t^9, t^{13}) = (t^5) : \mathfrak{m}$$

is weakly $\mathfrak{m}$-full, but not $\mathfrak{m}$-full.

Theorem 2.9 (Goto-Hayasaka, 2002)

Suppose that I is $\mathfrak{m}$-full and $\operatorname{depth} R/I = 0$. *If* $\operatorname{id}_R I < \infty$, *then R is a RLR.*

Theorem 2.10 (Celikbas-Iima-Sadeghi-Takahashi, 2018)

Suppose that I is weakly $\mathfrak{m}$-full and $\operatorname{depth} R/I = 0$. *If* $\operatorname{id}_R I < \infty$, *then R is a RLR.*

§3 Proof of Theorem 1.8

Setting 3.1

- $(R, \mathfrak{m})$ a Noetherian local ring
- I an $\mathfrak{m}$-primary ideal of R
- M a finitely generated R-module

Proposition 3.2 (cf. Corso-Huneke-Katz-Vasconcelos, 2006)

Suppose that I is weakly $\mathfrak{m}$-full. If

$$\mathrm{Tor}_t^R(M, R/I) = 0 \quad \textit{for } \exists\, t \geq 0$$

then $\mathrm{pd}_R\, M < t$.

Proof of Proposition 3.2

If $t = 0$, then $M = 0$ and $\operatorname{pd}_R M = -\infty$. Thus we may assume $t > 0$. Consider a minimal free resolution of M

$$\cdots \longrightarrow F_{t+1} \longrightarrow F_t \xrightarrow{\partial_t} F_{t-1} \to \cdots \to F_0 \to 0.$$

Applying $\overline{(-)} = (-) \otimes_R R/I$, we obtain

$$\cdots \longrightarrow \overline{F}_{t+1} \longrightarrow \overline{F}_t \xrightarrow{\overline{\partial}_t} \overline{F}_{t-1} \to \cdots \to \overline{F}_0 \to 0.$$

Suppose $\overline{\partial}_t = 0$. Then $\overline{F}_t = \mathfrak{m}\overline{F}_t$, whence $\overline{F}_t = (0)$ and $F_t = (0)$. Hence $\operatorname{pd}_R M < t$.

We now assume $\operatorname{Im} \overline{\partial}_t \neq 0$ and seek a contradiction. Since $(\operatorname{Im} \partial_t) I \subseteq IF_{t-1}$,

$$(\operatorname{Im} \partial_t)\mathfrak{m}^s \subseteq IF_{t-1} \quad \text{for} \quad \exists\, s > 0.$$

Let us choose the integer s as small as possible.

Proof of Proposition 3.2

Since $(\operatorname{Im}\partial_t)\mathfrak{m}^{s-1} \not\subseteq IF_{t-1}$, we choose

$$u \in (\operatorname{Im}\partial_t)\mathfrak{m}^{s-1} \ \text{ s.t. } \ u \notin IF_{t-1}.$$

Then

$$\mathfrak{m}u \subseteq \mathfrak{m}IF_{t-1}$$

and hence

$$u \in (\mathfrak{m}I : \mathfrak{m})F_{t-1} = IF_{t-1}$$

which is a contradiction. Therefore $\operatorname{Im}\overline{\partial}_t = 0$, as desired.

□

Corollary 3.3

Suppose that I is weakly $\mathfrak{m}$-full and depth $R > 0$. *Then*

$$I \otimes_R M \text{ is torsion-free} \iff M \text{ is free.}$$

Proof.

$(\Rightarrow)$ Applying $(-) \otimes_R M$ to $0 \to I \to R \to R/I \to 0$, we get

$$0 \longrightarrow \operatorname{Tor}_1^R(R/I, M) \longrightarrow I \otimes_R M.$$

Since I contains a NZD on R, $\operatorname{Tor}_1^R(R/I, M)$ is torsion. Hence

$$\operatorname{Tor}_1^R(R/I, M) = (0)$$

which implies $\operatorname{pd}_R M \leq 0$. □

We are now ready to prove Theorem 1.8.

Theorem 1.8 (Celikbas-Goto-Takahashi-T, 2018)

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with $\dim R = 1$, I an $\mathfrak{m}$-primary ideal of R. Suppose that I is weakly $\mathfrak{m}$-full. If $I \otimes_R I^*$ is torsion-free, then I is principal, and R is a DVR.

Proof of Theorem 1.8

Since $I \otimes_R I^*$ is torsion-free, by Corollary 3.3, I^* is free.

Claim 3.4

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with $\dim R = 1$, M a finitely generated torsion-free R-module. Then

$$M^* \text{ is free} \iff M \text{ is free}$$

Proof of Theorem 1.8

Proof of Claim 3.4.

We may assume M is indecomposable. Since M is torsion-free, if $M^* = (0)$, then $M = (0)$. We may assume $M^* \neq (0)$. Let

$$F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

be the minimal presentation of M. Then

$$0 \longrightarrow M^* \longrightarrow F_0^* \longrightarrow F_1^* \longrightarrow \operatorname{Tr} M \longrightarrow 0.$$

Since $\operatorname{pd}_R(\operatorname{Tr} M) < \infty$, we have $\operatorname{pd}_R(\operatorname{Tr} M) \leq 1$. Thus

$$\operatorname{Ext}_R^2(\operatorname{Tr} M, R) = (0)$$

which implies

$$M \longrightarrow M^{**} \longrightarrow 0$$

and hence $M \cong M^{**}$ as claimed. □

Proof of Theorem 1.8

Consequently, I is principal. Therefore, $I = (f)$ for $\exists$ R-NZD $f \in \mathfrak{m}$. Since $I \cong R$, we have $\mathrm{pd}_R(R/I) \leq 1$, so that

$$\mathrm{Tor}_2^R(R/\mathfrak{m}, R/I) = (0).$$

Hence, by Proposition 3.2, $\mathrm{pd}_R(R/\mathfrak{m}) \leq 1$, which yields that R is a DVR. □

Remark 3.5

(1) Let $(R, \mathfrak{m})$ be a Noetherian local ring with depth $R > 0$, I a weakly $\mathfrak{m}$-full $\mathfrak{m}$-primary ideal. If $\mu_R(I) = 1$, then R is a DVR.

(2) Assertion (1) holds for the case where $I = \overline{I}$.

§4 Integrally closed ideals

Theorem 4.1

Let R be a Noetherian ring, I an ideal of R with $\mathrm{ht}_R I > 0$. Assume R satisfies Serre's condition (S_2). Then TFAE.

(1) $I = \overline{I}$ *and* $[I] \in \mathrm{Pic}\, R$.

(2) $R_{\mathfrak{p}}$ *is a DVR for* $\forall \mathfrak{p} \in \mathrm{Ass}_R R/I$ *and* $[I] \in \mathrm{Pic}\, R$.

(3) $I = \overline{I}$ *and* $I \otimes_R I^*$ *is reflexive.*

When this is the case,

$$I = \bigcap_{\mathfrak{p} \in \mathrm{Ass}_R R/I} \mathfrak{p}^{(n(\mathfrak{p}))}$$

where $n(\mathfrak{p}) \geq 1$ *for* $\forall \mathfrak{p} \in \mathrm{Ass}_R R/I$ *and* $\mathfrak{p}^{(n(\mathfrak{p}))}$ *is a symbolic power of* $\mathfrak{p}$.

Recall that a Noetherian ring R satisfies (S_n) if

$$\operatorname{depth} R_{\mathfrak{p}} \geq \min\{n, \operatorname{ht}_R \mathfrak{p}\} \quad \text{for} \quad \forall \mathfrak{p} \in \operatorname{Spec} R.$$

We set

$$\operatorname{Pic} R = \{\text{finitely generated projective module } M \text{ s.t. } M_{\mathfrak{p}} \cong R_{\mathfrak{p}} \text{ for } \forall \mathfrak{p} \in \operatorname{Spec} R\}/\cong$$

and call it *the Picard group* of R.

Lemma 4.2

Let R be a Noetherian ring satisfying (S_1), I an ideal of R. Assume

$$IR_{\mathfrak{p}} \cong R_{\mathfrak{p}} \quad \textit{for} \quad \exists\, \mathfrak{p} \in \mathrm{Ass}_R\, R/I.$$

Then $R_{\mathfrak{p}}$ is a Cohen-Macaulay local ring with $\dim R_{\mathfrak{p}} = 1$. *If, furthermore, $I = \overline{I}$, then $R_{\mathfrak{p}}$ is a DVR.*

Proof.

Note that $\mathrm{depth}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/IR_{\mathfrak{p}}) = 0$. Since $IR_{\mathfrak{p}}$ is principal, we have $\mathrm{depth}\, R_{\mathfrak{p}} = 1$. As R satisfies (S_1), we conclude that $R_{\mathfrak{p}}$ is a Cohen-Macaulay local ring with $\dim R_{\mathfrak{p}} = 1$. If $I = \overline{I}$, then Remark 3.5 (2) shows $R_{\mathfrak{p}}$ is a DVR. □

Theorem 4.1

Let R be a Noetherian ring, satisfying (S_2), I an ideal of R with $\mathrm{ht}_R I > 0$. Then TFAE.

(1) $I = \overline{I}$ *and* $[I] \in \mathrm{Pic}\, R$.

(2) $R_\mathfrak{p}$ *is a DVR for* $\forall \mathfrak{p} \in \mathrm{Ass}_R R/I$ *and* $[I] \in \mathrm{Pic}\, R$.

(3) $I = \overline{I}$ *and* $I \otimes_R I^*$ *is reflexive.*

When this is the case,

$$I = \bigcap_{\mathfrak{p} \in \mathrm{Ass}_R R/I} \mathfrak{p}^{(n(\mathfrak{p}))}$$

where $n(\mathfrak{p}) \geq 1$ *for* $\forall \mathfrak{p} \in \mathrm{Ass}_R R/I$.

Proof of Theorem 4.1

(1) $\Rightarrow$ (2): It follows from Lemma 4.2.

(2) $\Rightarrow$ (3): Suppose that $\overline{I}/I \neq (0)$ and choose $\mathfrak{p} \in \operatorname{Ass}_R \overline{I}/I$. Then $\mathfrak{p} \in \operatorname{Ass}_R R/I$ so that $R_\mathfrak{p}$ is a DVR by assumption. Hence

$$IR_\mathfrak{p} = \overline{IR_\mathfrak{p}} = \overline{I}R_\mathfrak{p}.$$

This is a contradiction, because $(\overline{I}/I)_\mathfrak{p} \neq (0)$. Thus $\overline{I} = I$.

Since $[I] \in \operatorname{Pic} R$, I is projective.

Note that

- R-duals and tensor products of projective modules are projective
- projective modules are reflexive

so we conclude that $I \otimes_R I^*$ is reflexive.

(2) $\Rightarrow$ (1): It follows from the fact that (2) implies $I = \overline{I}$.

Proof of Theorem 4.1

(3) $\Rightarrow$ (2): Suppose that $I = \overline{I}$ and $I \otimes_R I^*$ is reflexive.

Claim 4.3

Let $\mathfrak{p} \in \operatorname{Spec} R$ *with* $\operatorname{ht}_R \mathfrak{p} \leq 1$. *Then* $IR_{\mathfrak{p}} \cong R_{\mathfrak{p}}$ *and* $\operatorname{Supp}_R I = \operatorname{Spec} R$.

We now proceed to show $[I] \in \operatorname{Pic} R$, by using Theorem 4.4.

Theorem 4.4 (Auslander, Huneke-Wiegand)

Let $(R, \mathfrak{m})$ *be a Noetherian local ring satisfying* (S_2), M *a finitely generated torsion-free* R*-module. Suppose that*

$$M_P \text{ is a free } R_P\text{-module for } P \in \operatorname{Spec} R \text{ with } \operatorname{ht}_R P \leq 1.$$

If $M \otimes_R M^*$ *is reflexive, then* M *is free.*

Proof of Theorem 4.1

For $\mathfrak{q} \in \operatorname{Spec} R$, let $P \in \operatorname{Spec} R_{\mathfrak{q}}$ with $\operatorname{ht}_{R_{\mathfrak{q}}} P \leq 1$.

Then $P = \mathfrak{p} R_{\mathfrak{q}}$ for $\exists\ \mathfrak{p} \in \operatorname{Spec} R$ with $\dim R_{\mathfrak{p}} \leq 1$. It follows that

$$(IR_{\mathfrak{q}})_P \cong IR_{\mathfrak{p}} \cong R_{\mathfrak{p}} \cong (R_{\mathfrak{q}})_P.$$

Moreover $IR_{\mathfrak{q}} \otimes_{R_{\mathfrak{q}}} (IR_{\mathfrak{q}})^*$ is reflexive. Theorem 4.4 implies $IR_{\mathfrak{q}}$ is $R_{\mathfrak{q}}$-free.

Since $\operatorname{Supp}_R I = \operatorname{Spec} R$, we see that $IR_{\mathfrak{q}} \cong R_{\mathfrak{q}}$.

This shows I is projective, i.e., $[I] \in \operatorname{Pic} R$.

Proof of Theorem 4.1

Let $\mathfrak{p} \in \operatorname{Ass}_R R/I$. Since $[I] \in \operatorname{Pic} R$, we have $IR_\mathfrak{p} \cong R_\mathfrak{p}$. By Lemma 4.2, $R_\mathfrak{p}$ is a Cohen-Macaulay ring with $\dim R_\mathfrak{p} = 1$. As $IR_\mathfrak{p}$ is free, we have

$$\operatorname{Tor}_2^{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, R_\mathfrak{p}/IR_\mathfrak{p}) = 0.$$

By Proposition 3.2, $R_\mathfrak{p}$ is a DVR. This completes the proof of (3) $\Rightarrow$ (2).

Let us make sure of the last assertion.

Let $\mathfrak{p} \in \operatorname{Ass}_R R/I$. Then, since $R_\mathfrak{p}$ is a DVR, we have

$$IR_\mathfrak{p} = \mathfrak{p}^{n(\mathfrak{p})} R_\mathfrak{p}$$

for $\exists\ n(\mathfrak{p}) \geq 1$. Thus $IR_\mathfrak{p} \cap R = \mathfrak{p}^{(n(\mathfrak{p}))}$, and hence the result follows. $\square$

Thank you so much for your attention.

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